DIFFERENTIATION RULES

$\frac{d}{dx}\Big(f(x)+g(x)\Big) = f'(x) + g'(x)$

$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{\big(f(x)\big)^2}$

$\frac{d}{dx}\Big(cf(x)\Big) = cf'(x)$

$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{\big(g(x)\big)^2}$

$\frac{d}{dx}\Big(f(x)g(x)\Big) = f'(x)g(x) + f(x)g'(x)$

$\frac{d}{dx}f\Big(g(x)\Big) = f'\Big(g(x)\Big)g'(x)$

ELEMENTARY DERIVATIVES

$\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$

$\frac{d}{dx}e^x = e^x$

$\frac{d}{dx}\sin x = \cos x$

$\frac{d}{dx}\sec x = \sec x \tan x$

$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

$\frac{d}{dx}a^x = a^x \ln a \quad (a > 0)$

$\frac{d}{dx}\cos x = -\sin x$

$\frac{d}{dx}\csc x = -\csc x \cot x$

$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$

$\frac{d}{dx}x^r = rx^{r-1}$

$\frac{d}{dx}\ln x = \frac{1}{x} \quad (x > 0)$

$\frac{d}{dx}\tan x = \sec^2 x$

$\frac{d}{dx}\cot x = -\csc^2 x$

$\frac{d}{dx}|x| = \operatorname{sgn} x = \frac{x}{|x|}$

TRIGONOMETRIC IDENTITIES

$\sin^2 x + \cos^2 x = 1$

$\sec^2 x = 1 + \tan^2 x$

$\csc^2 x = 1 + \cot^2 x$

$\sin(-x) = -\sin x$

$\sin(\pi - x) = \sin x$

$\sin\left(\frac{\pi}{2} - x\right) = \cos x$

$\cos(-x) = \cos x$

$\cos(\pi - x) = -\cos x$

$\cos\left(\frac{\pi}{2} - x\right) = \sin x$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1 - \cos 2x}{2}$

$\cos^2 x = \frac{1 + \cos 2x}{2}$

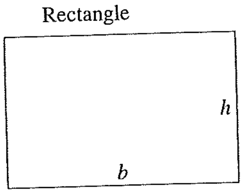
QUADRATIC FORMULA

If $Ax^2 + Bx + C = 0$, then $x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$.

GEOMETRIC FORMULAS

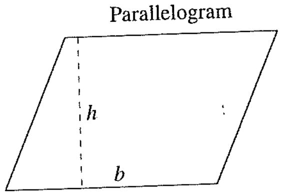
A = area, b = base, h = height, C = circumference, V = volume, S = surface area

Rectangle



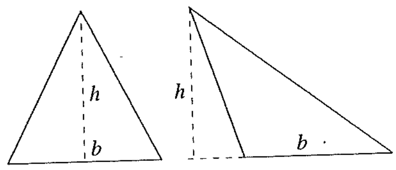
$A = bh$

Parallelogram



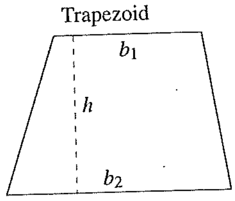
$A = bh$

Triangles



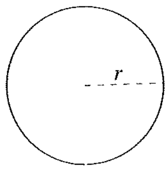
$A = \frac{1}{2}bh$

Trapezoid



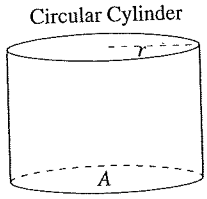
$A = \frac{1}{2}(b_1 + b_2)h$

Circle



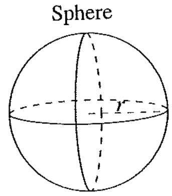
$A = \pi r^2,\ C = 2\pi r$

Circular Cylinder



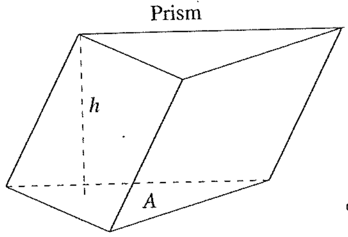
$V = Ah = \pi r^2 h,\ S = Ch = 2\pi rh$ (cylindrical wall)

Sphere



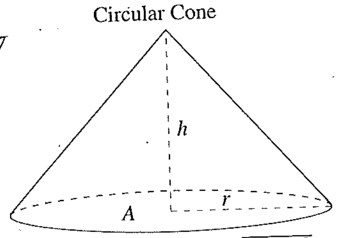
$V = \frac{4}{3}\pi r^3,\ S = 4\pi r^2$

Prism



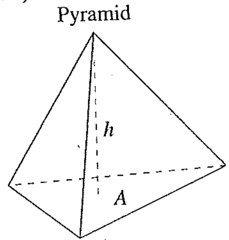
$V = Ah$

Circular Cone



$V = \frac{1}{3}Ah = \frac{1}{3}\pi r^2 h,\ S = \pi r\sqrt{r^2 + h^2}$ (conical wall)

Pyramid



$V = \frac{1}{3}Ah$

VECTOR IDENTITIES

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$ then

(dot product) $\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$

(cross product) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$

length of $\mathbf{u} = |\mathbf{u}| = \sqrt{\mathbf{u} \bullet \mathbf{u}} = \sqrt{u_1^2 + u_2^2 + u_3^2}$

angle between $\mathbf{u}$ and $\mathbf{v} = \cos^{-1}\left(\dfrac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$

triple product identities $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \bullet (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \bullet (\mathbf{u} \times \mathbf{v})$ $\quad \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \bullet \mathbf{w})\mathbf{v} - (\mathbf{u} \bullet \mathbf{v})\mathbf{w}$

IDENTITIES INVOLVING GRADIENT, DIVERGENCE, CURL, AND LAPLACIAN

$\nabla = \mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}$ ("del" or "nabla" operator)

$\nabla\phi(x, y, z) = \mathbf{grad}\,\phi(x, y, z) = \dfrac{\partial \phi}{\partial x}\mathbf{i} + \dfrac{\partial \phi}{\partial y}\mathbf{j} + \dfrac{\partial \phi}{\partial z}\mathbf{k}$

$\mathbf{F}(x, y, z) = F_1(x, y, z)\,\mathbf{i} + F_2(x, y, z)\,\mathbf{j} + F_3(x, y, z)\,\mathbf{k}$

$\nabla \bullet \mathbf{F}(x, y, z) = \mathbf{div}\,\mathbf{F}(x, y, z) = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$

$$\nabla \times \mathbf{F}(x, y, z) = \mathbf{curl}\,\mathbf{F}(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}\right)\mathbf{k}$$

$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

$\nabla \bullet (\phi\mathbf{F}) = (\nabla\phi) \bullet \mathbf{F} + \phi\,(\nabla \bullet \mathbf{F})$

$\nabla \times (\phi\mathbf{F}) = (\nabla\phi) \times \mathbf{F} + \phi\,(\nabla \times \mathbf{F})$

$\nabla \times (\nabla\phi) = \mathbf{0}$ $\quad$ (**curl grad** $= \mathbf{0}$)

$\nabla^2\phi(x, y, z) = \nabla \bullet \nabla\phi(x, y, z) = \mathbf{div}\,\mathbf{grad}\,\phi = \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + \dfrac{\partial^2 \phi}{\partial z^2}$

$\nabla \bullet (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \bullet \mathbf{G} - \mathbf{F} \bullet (\nabla \times \mathbf{G})$

$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \bullet \mathbf{G}) - \mathbf{G}(\nabla \bullet \mathbf{F}) - (\mathbf{F} \bullet \nabla)\mathbf{G} + (\mathbf{G} \bullet \nabla)\mathbf{F}$

$\nabla(\mathbf{F} \bullet \mathbf{G}) = \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \bullet \nabla)\mathbf{G} + (\mathbf{G} \bullet \nabla)\mathbf{F}$

$\nabla \bullet (\nabla \times \mathbf{F}) = 0$ $\quad$ (**div curl** $= 0$)

$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \bullet \mathbf{F}) - \nabla^2\mathbf{F}$ $\quad$ (**curl curl** $=$ **grad div** $-$ laplacian)

VERSIONS OF THE FUNDAMENTAL THEOREM OF CALCULUS

$\displaystyle\int_a^b f'(t)\,dt = f(b) - f(a)$ $\quad$ (the one-dimensional **Fundamental Theorem**)

$\displaystyle\int_C \mathbf{grad}\,\phi \bullet d\mathbf{r} = \phi\big(\mathbf{r}(b)\big) - \phi\big(\mathbf{r}(a)\big)$ if C is the curve $\mathbf{r} = \mathbf{r}(t)$, $(a \le t \le b)$.

$\displaystyle\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C F_1(x, y)\,dx + F_2(x, y)\,dy$ where C is the positively oriented boundary of R $\quad$ **(Green's Theorem)**

$\displaystyle\iint_S \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\,dS = \oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C F_1(x, y, z)\,dx + F_2(x, y, z)\,dy + F_3(x, y, z)\,dz$ where C is the oriented boundary of S. $\quad$ **(Stokes's Theorem)**

Three-dimensional versions: S is the closed boundary of D, with outward normal $\hat{\mathbf{N}}$

$\displaystyle\iiint_D \mathbf{div}\,\mathbf{F}\,dV = \oiint_S \mathbf{F} \bullet \hat{\mathbf{N}}\,dS$ $\quad$ **Divergence Theorem**

$\displaystyle\iiint_D \mathbf{curl}\,\mathbf{F}\,dV = -\oiint_S \mathbf{F} \times \hat{\mathbf{N}}\,dS$

$\displaystyle\iiint_D \mathbf{grad}\,\phi\,dV = \oiint_S \phi\hat{\mathbf{N}}\,dS$

FORMULAS RELATING TO CURVES IN 3-SPACE

Curve: $\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

Velocity: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = v\hat{\mathbf{T}}$

Speed: $v = |\mathbf{v}| = \dfrac{ds}{dt}$

Arc length: $s = \displaystyle\int_{t_0}^{t} v\,dt$

Acceleration: $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$

Tangential and normal components: $\mathbf{a} = \dfrac{dv}{dt}\hat{\mathbf{T}} + v^2\kappa\hat{\mathbf{N}}$

Unit tangent: $\hat{\mathbf{T}} = \dfrac{\mathbf{v}}{v}$

Binormal: $\hat{\mathbf{B}} = \dfrac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|}$

Normal: $\hat{\mathbf{N}} = \hat{\mathbf{B}} \times \hat{\mathbf{T}} = \dfrac{d\hat{\mathbf{T}}/dt}{|d\hat{\mathbf{T}}/dt|}$

Curvature: $\kappa = \dfrac{|\mathbf{v} \times \mathbf{a}|}{v^3}$

Radius of curvature: $\rho = \dfrac{1}{\kappa}$

Torsion: $\tau = \dfrac{(\mathbf{v} \times \mathbf{a}) \bullet (d\mathbf{a}/dt)}{|\mathbf{v} \times \mathbf{a}|^2}$

The Frenet-Serret formulas: $\dfrac{d\hat{\mathbf{T}}}{ds} = \kappa\hat{\mathbf{N}}$, $\quad \dfrac{d\hat{\mathbf{N}}}{ds} = -\kappa\hat{\mathbf{T}} + \tau\hat{\mathbf{B}}$, $\quad \dfrac{d\hat{\mathbf{B}}}{ds} = -\tau\hat{\mathbf{N}}$

ORTHOGONAL CURVILINEAR COORDINATES

transformation: $x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w)$

position vector: $\mathbf{r} = x(u, v, w)\mathbf{i} + y(u, v, w)\mathbf{j} + z(u, v, w)\mathbf{k}$

scale factors: $h_u = \left|\frac{\partial \mathbf{r}}{\partial u}\right|, \quad h_v = \left|\frac{\partial \mathbf{r}}{\partial v}\right|, \quad h_w = \left|\frac{\partial \mathbf{r}}{\partial w}\right|$

local basis: $\hat{\mathbf{u}} = \frac{1}{h_u}\frac{\partial \mathbf{r}}{\partial u}, \quad \hat{\mathbf{v}} = \frac{1}{h_v}\frac{\partial \mathbf{r}}{\partial v}, \quad \hat{\mathbf{w}} = \frac{1}{h_w}\frac{\partial \mathbf{r}}{\partial w}$

volume element: $dV = h_u h_v h_w \, du\, dv\, dw$

scalar field: $f(u, v, w)$

vector field: $\mathbf{F}(u, v, w) = F_u(u, v, w)\hat{\mathbf{u}} + F_v(u, v, w)\hat{\mathbf{v}} + F_w(u, v, w)\hat{\mathbf{w}}$

gradient: $\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{\mathbf{u}} + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{\mathbf{v}} + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{\mathbf{w}}$

divergence: $\nabla \bullet \mathbf{F} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(h_v h_w F_u\right) + \frac{\partial}{\partial v}\left(h_u h_w F_v\right) + \frac{\partial}{\partial w}\left(h_u h_v F_w\right)\right]$

$$\nabla^2 f = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial f}{\partial w}\right)\right]$$

curl: $\nabla \times \mathbf{F} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{\mathbf{u}} & h_v\hat{\mathbf{v}} & h_w\hat{\mathbf{w}} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ F_u h_u & F_v h_v & F_w h_w \end{vmatrix}$

PLANE POLAR COORDINATES

transformation: $x = r\cos\theta, \quad y = r\sin\theta$

position vector: $\mathbf{r} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$

scale factors: $h_r = \left|\frac{\partial \mathbf{r}}{\partial r}\right| = 1, \quad h_\theta = \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = r$

local basis: $\hat{\mathbf{r}} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \quad \hat{\boldsymbol{\theta}} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$

area element: $dA = r\,dr\,d\theta$

scalar field: $f(r, \theta)$

vector field: $\mathbf{F}(r, \theta) = F_r(r, \theta)\hat{\mathbf{r}} + F_\theta(r, \theta)\hat{\boldsymbol{\theta}}$

gradient: $\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}}$

divergence: $\nabla \bullet \mathbf{F} = \frac{\partial F_r}{\partial r} + \frac{1}{r}F_r + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}$

laplacian: $\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$

curl: $\nabla \times \mathbf{F} = \left[\frac{\partial F_\theta}{\partial r} + \frac{F_\theta}{r} - \frac{1}{r}\frac{\partial F_r}{\partial \theta}\right]\mathbf{k}$

CYLINDRICAL COORDINATES

transformation: $x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$

position vector: $\mathbf{r} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + z\mathbf{k}$

scale factors: $h_r = \left|\frac{\partial \mathbf{r}}{\partial r}\right| = 1, \quad h_\theta = \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = r, \quad h_z = \left|\frac{\partial \mathbf{r}}{\partial z}\right| = 1$

local basis: $\hat{\mathbf{r}} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \quad \hat{\boldsymbol{\theta}} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}, \quad \hat{\mathbf{z}} = \mathbf{k}$

volume element: $dV = r\,dr\,d\theta\,dz$

surface area element (on $r = a$): $dS = a\,d\theta\,dz$

scalar field: $f(r, \theta, z)$

vector field: $\mathbf{F}(r, \theta, z) = F_r(r, \theta, z)\hat{\mathbf{r}} + F_\theta(r, \theta, z)\hat{\boldsymbol{\theta}} + F_z(r, \theta, z)\mathbf{k}$

gradient: $\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\mathbf{k}$

divergence: $\nabla \bullet \mathbf{F} = \frac{\partial F_r}{\partial r} + \frac{1}{r}F_r + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$

laplacian: $\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$

curl: $\nabla \times \mathbf{F} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}$

SPHERICAL COORDINATES

transformation: $x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$

position vector: $\mathbf{r} = \rho\sin\phi\cos\theta\,\mathbf{i} + \rho\sin\phi\sin\theta\,\mathbf{j} + \rho\cos\phi\mathbf{k}$

scale factors: $h_\rho = \left|\frac{\partial \mathbf{r}}{\partial \rho}\right| = 1, \quad h_\phi = \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = \rho, \quad h_\theta = \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = \rho\sin\phi$

local basis: $\hat{\boldsymbol{\rho}} = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}, \quad \hat{\boldsymbol{\phi}} = \cos\phi\cos\theta\,\mathbf{i} + \cos\phi\sin\theta\,\mathbf{j} - \sin\phi\,\mathbf{k}, \quad \hat{\boldsymbol{\theta}} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$

volume element: $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

surface area element (on $\rho = a$): $dS = a^2\sin\phi\,d\theta\,d\phi$

scalar field: $f(\rho, \phi, \theta)$

vector field: $\mathbf{F}(\rho, \phi, \theta) = F_\rho(\rho, \phi, \theta)\hat{\boldsymbol{\rho}} + F_\phi(\rho, \phi, \theta)\hat{\boldsymbol{\phi}} + F_\theta(\rho, \phi, \theta)\hat{\boldsymbol{\theta}}$

gradient: $\nabla f = \frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{1}{\rho\sin\phi}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}}$

divergence: $\nabla \bullet \mathbf{F} = \frac{\partial F_\rho}{\partial \rho} + \frac{2}{\rho}F_\rho + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\cot\phi}{\rho}F_\phi + \frac{1}{\rho\sin\phi}\frac{\partial F_\theta}{\partial \theta}$

laplacian: $\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\cot\phi}{\rho^2}\frac{\partial f}{\partial \phi} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}$

curl: $\nabla \times \mathbf{F} = \frac{1}{\rho^2\sin\phi}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \rho\sin\phi\hat{\boldsymbol{\theta}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ F_\rho & \rho F_\phi & \rho\sin\phi F_\theta \end{vmatrix}$

INTEGRATION RULES

$$\int (Af(x) + Bg(x))\,dx = A\int f(x)\,dx + B\int g(x)\,dx$$

$$\int f'(g(x))\,g'(x)\,dx = f(g(x)) + C$$

$$\int U(x)\,dV(x) = U(x)\,V(x) - \int V(x)\,dU(x)$$

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

ELEMENTARY INTEGRALS

$$\int x^r\,dx = \frac{1}{r+1}x^{r+1} + C \text{ if } r \neq -1$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C \quad (a > 0,\ |x| < a)$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \quad (a > 0)$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C \quad (a > 0)$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C \quad (a > 0,\ |x| > a)$$

TRIGONOMETRIC INTEGRALS

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$$

$$\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C$$

$$\int \tan^2 x\,dx = \tan x - x + C$$

$$\int \cot^2 x\,dx = -\cot x - x + C$$

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

$$\int \csc^3 x\,dx = -\frac{1}{2}\csc x \cot x + \frac{1}{2}\ln|\csc x - \cot x| + C$$

$$\int \sin ax\,\sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} + C \text{ if } a^2 \neq b^2$$

$$\int \cos ax\,\cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} + C \text{ if } a^2 \neq b^2$$

$$\int \sin ax\,\cos bx\,dx = -\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)} + C \text{ if } a^2 \neq b^2$$

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\,\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\,\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx \text{ if } n \neq 1$$

$$\int \cot^n x\,dx = \frac{-1}{n-1}\cot^{n-1} x - \int \cot^{n-2} x\,dx \text{ if } n \neq 1$$

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x\,\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx \text{ if } n \neq 1$$

$$\int \csc^n x\,dx = \frac{-1}{n-1}\csc^{n-2} x\,\cot x + \frac{n-2}{n-1}\int \csc^{n-2} x\,dx \text{ if } n \neq 1$$

$$\int \sin^n x\,\cos^m x\,dx = -\frac{\sin^{n-1} x\,\cos^{m+1} x}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} x\,\cos^m x\,dx \text{ if } n \neq -m$$

$$\int \sin^n x\,\cos^m x\,dx = \frac{\sin^{n+1} x\,\cos^{m-1} x}{n+m} + \frac{m-1}{n+m}\int \sin^n x\,\cos^{m-2} x\,dx \text{ if } m \neq -n$$

$$\int x\,\sin x\,dx = \sin x - x\,\cos x + C$$

$$\int x\,\cos x\,dx = \cos x + x\,\sin x + C$$

$$\int x^n\,\sin x\,dx = -x^n\,\cos x + n\int x^{n-1}\cos x\,dx$$

$$\int x^n\,\cos x\,dx = x^n\,\sin x - n\int x^{n-1}\sin x\,dx$$

INTEGRALS INVOLVING $\sqrt{x^2 \pm a^2}$ $(a > 0)$

(If $\sqrt{x^2 - a^2}$, assume $x > a > 0$.)

$$\int \sqrt{x^2 \pm a^2}\, dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \ln|x + \sqrt{x^2 \pm a^2}| + C$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}| + C$$

$$\int \frac{\sqrt{x^2 + a^2}}{x}\, dx = \sqrt{x^2 + a^2} - a \ln\left|\frac{a + \sqrt{x^2 + a^2}}{x}\right| + C$$

$$\int \frac{\sqrt{x^2 - a^2}}{x}\, dx = \sqrt{x^2 - a^2} - a \tan^{-1}\frac{\sqrt{x^2 - a^2}}{a} + C$$

$$\int x^2\sqrt{x^2 \pm a^2}\, dx = \frac{x}{8}(2x^2 \pm a^2)\sqrt{x^2 \pm a^2} - \frac{a^4}{8} \ln|x + \sqrt{x^2 \pm a^2}| + C$$

$$\int \frac{x^2}{\sqrt{x^2 \pm a^2}}\, dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \mp \frac{a^2}{2} \ln|x + \sqrt{x^2 \pm a^2}| + C$$

$$\int \frac{\sqrt{x^2 \pm a^2}}{x^2}\, dx = -\frac{\sqrt{x^2 \pm a^2}}{x} + \ln|x + \sqrt{x^2 \pm a^2}| + C$$

$$\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp\frac{\sqrt{x^2 \pm a^2}}{a^2 x} + C$$

$$\int \frac{dx}{(x^2 \pm a^2)^{3/2}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}} + C$$

$$\int (x^2 \pm a^2)^{3/2}\, dx = \frac{x}{8}(2x^2 \pm 5a^2)\sqrt{x^2 \pm a^2} + \frac{3a^4}{8} \ln|x + \sqrt{x^2 \pm a^2}| + C$$

INTEGRALS INVOLVING $\sqrt{a^2 - x^2}$ $(a > 0, |x| < a)$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{\sqrt{a^2 - x^2}}{x}\, dx = \sqrt{a^2 - x^2} - a \ln\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right| + C$$

$$\int \frac{x^2}{\sqrt{a^2 - x^2}}\, dx = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + C$$

$$\int x^2\sqrt{a^2 - x^2}\, dx = \frac{x}{8}(2x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{8} \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{x^2\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x} + C$$

$$\int \frac{\sqrt{a^2 - x^2}}{x^2}\, dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right| + C$$

$$\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 - x^2}} + C$$

$$\int (a^2 - x^2)^{3/2}\, dx = \frac{x}{8}(5a^2 - 2x^2)\sqrt{a^2 - x^2} + \frac{3a^4}{8} \sin^{-1}\frac{x}{a} + C$$

INTEGRALS OF INVERSE TRIGONOMETRIC FUNCTIONS

$$\int \sin^{-1}x\, dx = x \sin^{-1}x + \sqrt{1 - x^2} + C$$

$$\int \tan^{-1}x\, dx = x \tan^{-1}x - \frac{1}{2} \ln(1 + x^2) + C$$

$$\int \sec^{-1}x\, dx = x \sec^{-1}x - \ln|x + \sqrt{x^2 - 1}| + C \quad (x > 1)$$

$$\int x \sin^{-1}x\, dx = \frac{1}{4}(2x^2 - 1)\sin^{-1}x + \frac{x}{4}\sqrt{1 - x^2} + C$$

$$\int x \tan^{-1}x\, dx = \frac{1}{2}(x^2 + 1)\tan^{-1}x - \frac{x}{2} + C$$

$$\int x \sec^{-1}x\, dx = \frac{x^2}{2} \sec^{-1}x - \frac{1}{2}\sqrt{x^2 - 1} + C \quad (x > 1)$$

$$\int x^n \sin^{-1}x\, dx = \frac{x^{n+1}}{n+1}\sin^{-1}x - \frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{1 - x^2}}\, dx + C \text{ if } n \neq -1$$

$$\int x^n \tan^{-1}x\, dx = \frac{x^{n+1}}{n+1}\tan^{-1}x - \frac{1}{n+1}\int \frac{x^{n+1}}{1 + x^2}\, dx + C \text{ if } n \neq -1$$

$$\int x^n \sec^{-1}x\, dx = \frac{x^{n+1}}{n+1}\sec^{-1}x - \frac{1}{n+1}\int \frac{x^n}{\sqrt{x^2 - 1}}\, dx + C \quad (n \neq -1, x >$$

EXPONENTIAL AND LOGARITHMIC INTEGRALS

$$\int x e^x\, dx = (x - 1)e^x + C$$

$$\int x^n e^x\, dx = x^n e^x - n \int x^{n-1} e^x\, dx$$

$$\int \ln x\, dx = x \ln x - x + C$$

$$\int x^n \ln x\, dx = \frac{x^{n+1}}{n+1} \ln x - \frac{x^{n+1}}{(n+1)^2} + C, \quad (n \neq -1)$$

$$\int x^n (\ln x)^m\, dx = \frac{x^{n+1}}{n+1}(\ln x)^m - \frac{m}{n+1}\int x^n (\ln x)^{m-1}\, dx \quad (n \neq -1)$$

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + C$$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + C$$

INTEGRALS OF HYPERBOLIC FUNCTIONS

$$\int \sinh x\, dx = \cosh x + C$$

$$\int \cosh x\, dx = \sinh x + C$$

$$\int \tanh x\, dx = \ln(\cosh x) + C$$

$$\int \coth x\, dx = \ln|\sinh x| + C$$

$$\int \operatorname{sech} x\, dx = 2\tan^{-1}(e^x) + C$$

$$\int \operatorname{csch} x\, dx = \ln\left|\tanh\frac{x}{2}\right| + C$$

$$\int \sinh^2 x\, dx = \frac{1}{4} \sinh 2x - \frac{x}{2} + C$$

$$\int \cosh^2 x\, dx = \frac{1}{4} \sinh 2x + \frac{x}{2} + C$$

$$\int \tanh^2 x\, dx = x - \tanh x + C$$

$$\int \coth^2 x\, dx = x - \coth x + C$$

$$\int \operatorname{sech}^2 x\, dx = \tanh x + C$$

$$\int \operatorname{csch}^2 x\, dx = -\coth x + C$$

$$\int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x + C$$

$$\int \operatorname{csch} x \coth x\, dx = -\operatorname{csch} x + C$$

MISCELLANEOUS ALGEBRAIC INTEGRALS

$$\int x(ax+b)^{-1}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C$$

$$\int x(ax+b)^{-2}\,dx = \frac{1}{a^2}\left[\ln|ax+b| + \frac{b}{ax+b}\right] + C$$

$$\int x(ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a^2}\left(\frac{ax+b}{n+2} - \frac{b}{n+1}\right) + C \text{ if } n \neq -1,\ -2$$

$$\int \frac{dx}{(a^2 \pm x^2)^n} = \frac{1}{2a^2(n-1)}\left(\frac{x}{(a^2 \pm x^2)^{n-1}} + (2n-3)\int \frac{dx}{(a^2 \pm x^2)^{n-1}}\right) \text{ if } n \neq 1$$

$$\int x\sqrt{ax+b}\,dx = \frac{2}{15a^2}(3ax-2b)(ax+b)^{3/2} + C$$

$$\int x^n\sqrt{ax+b}\,dx = \frac{2}{a(2n+3)}\left(x^n(ax+b)^{3/2} - nb\int x^{n-1}\sqrt{ax+b}\,dx\right)$$

$$\int \frac{x\,dx}{\sqrt{ax+b}} = \frac{2}{3a^2}(ax-2b)\sqrt{ax+b} + C$$

$$\int \frac{x^n\,dx}{\sqrt{ax+b}} = \frac{2}{a(2n+1)}\left(x^n\sqrt{ax+b} - nb\int \frac{x^{n-1}}{\sqrt{ax+b}}\,dx\right)$$

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C \text{ if } b > 0$$

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{2}{\sqrt{-b}}\tan^{-1}\sqrt{\frac{ax+b}{-b}} + C \text{ if } b < 0$$

$$\int \frac{dx}{x^n\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{b(n-1)x^{n-1}} - \frac{(2n-3)a}{(2n-2)b}\int \frac{dx}{x^{n-1}\sqrt{ax+b}} \text{ if } n \neq 1$$

$$\int \sqrt{2ax-x^2}\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a} + C \quad (a > 0)$$

$$\int \frac{dx}{\sqrt{2ax-x^2}} = \sin^{-1}\frac{x-a}{a} + C \quad (a > 0)$$

$$\int x^n\sqrt{2ax-x^2}\,dx = -\frac{x^{n-1}(2ax-x^2)^{3/2}}{n+2} + \frac{(2n+1)a}{n+2}\int x^{n-1}\sqrt{2ax-x^2}\,dx$$

$$\int \frac{x^n\,dx}{\sqrt{2ax-x^2}} = -\frac{x^{n-1}}{n}\sqrt{2ax-x^2} + \frac{(2n-1)a}{n}\int \frac{x^{n-1}\,dx}{\sqrt{2ax-x^2}}$$

$$\int \frac{\sqrt{2ax-x^2}}{x}\,dx = \sqrt{2ax-x^2} + a\sin^{-1}\frac{x-a}{a} + C \quad (a > 0)$$

$$\int \frac{\sqrt{2ax-x^2}}{x^n}\,dx = \frac{(2ax-x^2)^{3/2}}{(3-2n)ax^n} + \frac{n-3}{(2n-3)a}\int \frac{\sqrt{2ax-x^2}}{x^{n-1}}\,dx$$

$$\int \frac{dx}{x^n\sqrt{2ax-x^2}} = \frac{\sqrt{2ax-x^2}}{a(1-2n)x^n} + \frac{n-1}{(2n-1)a}\int \frac{dx}{x^{n-1}\sqrt{2ax-x^2}}$$

$$\int (\sqrt{2ax-x^2})^n\,dx = \frac{x-a}{n+1}(\sqrt{2ax-x^2})^n + \frac{na^2}{n+1}\int (\sqrt{2ax-x^2})^{n-2}\,dx \text{ if } n \neq -1$$

$$\int \frac{dx}{(\sqrt{2ax-x^2})^n} = \frac{x-a}{(n-2)a^2}(\sqrt{2ax-x^2})^{2-n} + \frac{n-3}{(n-2)a^2}\int \frac{dx}{(\sqrt{2ax-x^2})^{n-2}} \text{ if } n \neq 2$$

DEFINITE INTEGRALS

$$\int_0^\infty x^n e^{-x}\,dx = n!\ (n \geq 0)$$

$$\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\ a > 0$$

$$\int_0^\infty xe^{-ax^2}\,dx = \frac{1}{2a} \text{ if } a > 0$$

$$\int_0^\infty x^n e^{-ax^2}\,dx = \frac{n-1}{2a}\int_0^\infty x^{n-2}e^{-ax^2}\,dx \text{ if } a > 0,\ n \geq 2$$

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}\dfrac{\pi}{2} & \text{if } n \text{ is an even integer and } n \geq 2 \\ \dfrac{2\cdot 4\cdot 6\cdots(n-1)}{3\cdot 5\cdot 7\cdots n} & \text{if } n \text{ is an odd integer and } n \geq 3 \end{cases}$$